Brownian motion and Stochastic Calculus
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Assignment 11

Exercise 1

Let B be an $(\mathbb{F},\mathbb{P})$–Brownian motion and M an $(\mathbb{F},\mathbb{P})$-martingale such that $\mathrm{d}M_t = \sigma M_t \mathrm{d}B_t$ with $\sigma > 0$ given and $M_0 = 1$.

1) Give the Itô decomposition of $Y_t := (M_t)^{-1}$, $t \geq 0$.

2) Let $\mathbb{Q}$ be the probability measure defined by $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} := M$. What can you say about the law of Y under $\mathbb{Q}$?

3) Let $K \geq 0$ be given. Show that

$$\mathbb{E}^{\mathbb{P}}\big[(M_T - K)^+\big] = K\mathbb{E}^{\mathbb{P}}\bigg[\bigg(\frac{1}{K} - M_T\bigg)^+\bigg].$$

Exercise 2

The goal of this question is to prove Novikov's condition which gives a sufficient requirement for an exponential (local) martingale to be a uniformly integrable martingale. Let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ be a filtration satisfying the usual conditions. When N is a continuous $(\mathbb{F},\mathbb{P})$–local martingale we will write $\mathcal{E}(N) := \exp(N - [N]/2)$. Suppose that M is a given continuous $(\mathbb{F},\mathbb{P})$–local martingale with $M_0 = 0$, $\mathbb{P}$–a.s.

1) Prove that $\mathcal{E}(M)$ is an $(\mathbb{F},\mathbb{P})$–super-martingale. Moreover, show that if $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty] = 1$, then $\mathcal{E}(M)$ is a $\mathbb{P}$–uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale.

2) Consider $p \geq 1$, $\varepsilon \in (0,1)$, $\eta \in (0,1)$ and $\rho \in \mathbb{R}$. Prove that when M is bounded by a deterministic constant, then for any $t \geq 0$

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{s\in[0,\infty)} \mathcal{E}(\eta M)_s^p\bigg] \leq \bigg(\frac{p}{p-1}\bigg)^p \sup_{s\in[0,\infty)} \mathbb{E}^{\mathbb{P}}\big[\mathcal{E}(\eta M)_s^p\big], \text{ for } p > 1,$$
$$\mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{e}^{\eta M_t - [\eta M]_t/2}\big)^p\Big] \leq \mathbb{E}^{\mathbb{P}}\Big[\big(\mathrm{e}^{(\rho-p)[\eta M]_t/2}\big)^{1/\varepsilon}\Big]^\varepsilon, \text{ for } \rho = p^2/(1-\varepsilon).$$

3) Use 2) and a localisation argument to establish the following: If $\mathbb{E}^{\mathbb{P}}[\mathrm{e}^{[M]_\infty/2}] < +\infty$ (called Novikov's condition) then for all $\eta \in (0,1)$, there exists $p > 1$ such that

$$\mathbb{E}^{\mathbb{P}}\bigg[\sup_{t\in[0,\infty)} \mathcal{E}(\eta M)^p\bigg] < \infty, \text{ and hence } \mathbb{E}^{\mathbb{P}}\bigg[\sup_{t\in[0,\infty)} \mathcal{E}(\eta M)\bigg] < +\infty.$$

Deduce that then $\mathcal{E}(\eta M)$ is a $\mathbb{P}$–uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale, so that $\mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_t] = 1$, for all $t \in [0,\infty]$.

4) Using 3) and part of the argument given in 2), show that (again assuming Novikov's condition) for $\varepsilon \in (0,1)$

$$1 = \mathbb{E}^{\mathbb{P}}[\mathcal{E}(\eta M)_\infty] \leq \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{(1-\varepsilon)[M]_\infty/2}\big]^\varepsilon \leq \mathbb{E}^{\mathbb{P}}[\mathcal{E}(M)_\infty]^{1-\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{[M]_\infty/2}\big]^\varepsilon, \text{ where } \eta := 1-\varepsilon.$$

5) Combine the above results to deduce that under Novikov's condition, $\mathcal{E}(M)$ is a $\mathbb{P}$–uniformly integrable $(\mathbb{F},\mathbb{P})$-martingale.

Exercise 3

Let B be a standard Brownian motion (in some filtration satisfying the usual conditions). Fix $t \geq 0$. The goal of this exercise is to compute the moment generating function of $\int_0^t B_s^2 \mathrm{d}s$. To this end, fix $\kappa > 0$ and $t \geq 0$.

1) Show that the process D defined by

$$D_s^t := \exp\bigg(- \kappa \int_0^{s\wedge t} B_u \mathrm{d}B_u - \frac{\kappa^2}{2} \int_0^{s\wedge t} B_u^2 \mathrm{d}u \bigg),\ s \geq 0,$$

is a $\mathbb{P}$–uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale. Moreover, observe that $\int_0^s B_u \mathrm{d}B_u = (B_s^2 - s)/2$, $\mathbb{P}$–a.s., for all $s \geq 0$.

2) Now we define a new probability measure $\mathbb{Q}$ via $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} := D_\infty^t$. Prove that under the measure $\mathbb{Q}$, the process

$$W_s^t := B_s + \kappa \int_0^{s\wedge t} B_u \mathrm{d}u,$$

is a standard Brownian motion (in the given filtration). Deduce that under $\mathbb{Q}$, $B_t \sim N\big(0, (1 - \mathrm{e}^{-2\kappa t})/(2\kappa)\big)$. Use this to prove that

$$\mathbb{E}^{\mathbb{P}}\Big[\mathrm{e}^{-\frac{\kappa^2}{2}\int_0^t B_u^2 \mathrm{d}u}\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\mathrm{e}^{\frac{\kappa}{2}(B_t^2 - t)}\Big] = \frac{1}{\sqrt{\cosh(\kappa t)}}.$$

3) Let $\widetilde{B}$ be another standard Brownian motion, independent of B. Show that

$$\int_0^t \big(B_u^2 + \widetilde{B}_u^2\big)\mathrm{d}u \overset{\text{law}}{=} \inf\{s \geq 0 \colon |B_s| = t\}.$$

Is it true that $\int_0^\cdot \big(B_u^2 + \widetilde{B}_u^2\big)\mathrm{d}u \overset{\text{law}}{=} \inf\{s \geq 0 \colon |B_s| = \cdot\}$?

Exercise 4

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Let B be an $(\mathbb{F}, \mathbb{P})$–Brownian motion, μ a bounded $\mathbb{F}$-adapted and measurable process, and fix some $x_0 \in \mathbb{R}$.

1) Show that there exists a unique solution to the SDE

$$X_t = x_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t X_s \mathrm{d}B_s,\ t \geq 0,$$

which is given by

$$X_t = \mathcal{E}(B)_t \bigg(x_0 + \int_0^t \mathcal{E}(B)_s^{-1} \mu_s \mathrm{d}s \bigg),\ t \geq 0.$$

In particular, if $x_0 \geq 0$ and μ is valued in $\mathbb{R}_+$, show that X is also valued in $\mathbb{R}_+$.

2) Fix now $(x_1, x_2) \in \mathbb{R}^2$, as well as two maps a_1 and a_2 from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$ which are Lipschitz continuous and with linear growth with respect to their second variable, uniformly in the first one. Assume that $a_1 \geq a_2$ and $x_1 \geq x_2$. Show that there are unique solutions to the SDEs

$$X_t^i = x_i + \int_0^t a_i(s, X_s^i)\mathrm{d}s + \int_0^t X_s^i \mathrm{d}B_s,\ t \geq 0,\ i \in \{1, 2\},$$

and that $X^1 \geq X^2$.